

Log-Optimal Portfolio Selection Using the Blackwell Approachability Theorem

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Abstract. The goal of this paper is to apply game-theoretic methods to the classical problem of an optimal portfolio construction. We present a method for constructing a log-optimal portfolio using the well-calibrated forecasts of return vectors. Dawid's notion of calibration and the Blackwell approachability theorem are used for computing the well-calibrated forecasts. We select a portfolio using this "artificial" probability distribution of return vectors. Our portfolio performs asymptotically at least as well as any stationary portfolio that redistributes the investment at each round using a continuous function of side information. Unlike in classical mathematical finance theory, no stochastic assumptions are made about return vectors.

1 Introduction

The model of stock market considered in this paper is the one studied, among others, by Breiman [5], Algoet and Cover [2], and Cover [8]; see also Györfy et al. [15].

Consider an investor who can access k financial instruments (asset, bond, cash, return of a game, etc.), and who can rebalance his wealth in each round according to a portfolio vector $\mathbf{b} = (b(1), \dots, b(k))$. The evolution of the market in time is represented by a sequence of return vectors (market values) $\mathbf{x}_1, \mathbf{x}_2, \dots$. A return vector $\mathbf{x} = (x(1), \dots, x(k))$ is a vector of k nonnegative numbers representing price relatives for a given trading period. That is, the j th component $x(j)$ of \mathbf{x} expresses the ratio of the closing and opening prices of asset j . In other words, $x(j)$ is the factor by which capital invested in the j th asset grows during the trading period. We suppose that these components are bounded $x(j) \in [\lambda_1, \lambda_2]$ for all $1 \leq j \leq k$, where $0 < \lambda_1 < \lambda_2 < \infty$.

We assume that the assets are arbitrarily divisible, and they are available for buying or for selling in unbounded quantities at the current price at any given trading period; there are no transaction costs. The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector $\mathbf{b} = (b(1), \dots, b(k))$. The j th component $b(j)$ of the vector \mathbf{b} denotes the proportion of the investors capital invested in asset j . We assume that $b(j) \geq 0$ for all j and $\sum_{j=1}^k b(j) = 1$. This means that the investment strategy

is self-financing and consumption of capital is excluded. The nonnegativity of the components of \mathbf{b} means that short selling and buying stocks on margin are not permitted. Let Γ denotes the simplex of portfolio vectors \mathbf{b} .

Let S_0 denote investor's initial capital. Then, at the end of the first trading period, investor's wealth becomes

$$S_1 = S_0(\mathbf{b} \cdot \mathbf{x}) = S_0 \sum_{j=1}^k b(j)x(j),$$

where " \cdot " denotes the inner product.

Starting with an initial wealth S_0 , after T trading periods, an "investment strategy" $\mathbf{b}_1, \dots, \mathbf{b}_T$ achieves a wealth

$$S_T = S_0 \prod_{t=1}^T (\mathbf{b}_t \cdot \mathbf{x}_t).$$

For simplicity, in what follows we assume that $S_0 = 1$.

In modeling the behavior of the evolution of the market, two main approaches have been considered in the theory of sequential investment. In the probabilistic approach, we assume that return vectors \mathbf{x}_t are realization of a sequence of random process \mathbf{X}_t , where $t = 1, 2, \dots$, and describe a statistical model.

If a market process \mathbf{X}_t is memoryless, i.e., it is a sequence of independent and identically distributed (i.i.d.) random return vectors then it was shown by Morvai [19] that the portfolio¹ $\mathbf{b}^* = \operatorname{argmax}_{\mathbf{b}} E(\log(\mathbf{b} \cdot \mathbf{X}_1))$ is asymptotically optimal in the following sense: for any portfolio vector \mathbf{b} with finite² $E((\log(\mathbf{b} \cdot \mathbf{X}_1))^2)$, a condition of optimality $\liminf_{T \rightarrow \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq 0$ holds almost surely, where $S_T^* = \prod_{t=1}^T (\mathbf{b}^* \cdot \mathbf{X}_t)$ and $S_T = \prod_{t=1}^T (\mathbf{b} \cdot \mathbf{X}_t)$.

But i.i.d. model is insufficient if the return vectors of different trading periods have a statistical dependence, which seems to be the case in real-world markets. In general case, we consider an arbitrary random process $\mathbf{X}_1, \mathbf{X}_2, \dots$ generating return vectors $\mathbf{x}_1, \mathbf{x}_2, \dots$.

Algoet [1] and Cover [2] have constructed so-called log-optimum portfolio strategy. Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be an arbitrary stationary and ergodic process. Denote $\mathbf{X}_1^{t-1} = \mathbf{X}_1, \dots, \mathbf{X}_{t-1}$. A function $\mathbf{b}^*(\cdot)$ is said to be a log-optimal portfolio strategy if

$$E(\log(\mathbf{b}^*(\mathbf{X}_1^{t-1}) \cdot \mathbf{X}_t)) = \max_{\mathbf{b}(\cdot)} E(\log(\mathbf{b}(\mathbf{X}_1^{t-1}) \cdot \mathbf{X}_t)) \quad (1)$$

for all t .

Let S_T^* denote a wealth achieved by a log-optimum portfolio strategy $\mathbf{b}^*(\cdot)$ after T trading periods. Then for any stationary and ergodic process \mathbf{X}_t and for any other investment strategy $\mathbf{b}(\cdot)$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq 0 \quad (2)$$

¹ In what follows log denotes logarithm on the base 2

² That is true when X_i are uniformly bounded.

almost surely. Such a strategy is also called universal with respect to the process $\mathbf{X}_1, \mathbf{X}_2, \dots$.

Moreover, Algoet [1] and Györfy and Schäfer [14] have shown that there exists a strategy uniformly universal with respect to the class of all stationary and ergodic processes. This means that a strategy $\mathbf{b}^*(\cdot)$ exists such that for any stationary and ergodic process $\mathbf{X}_1, \mathbf{X}_2, \dots$ asymptotic inequality (2) holds almost surely. Györfy and Schäfer called this scheme a histogram-based investment strategy. Györfy [15] extended this result to the kernel-based case. The rate of convergence has not been studied in these papers.

Another, “worst-case”, approach allows a return sequence $\mathbf{x}_1, \mathbf{x}_2, \dots$ to take completely arbitrary values, and no stochastic model is imposed on the mechanism generating the price relatives. This approach was pioneered by Cover [8]. Cover has shown that there exists an investment strategy \mathbf{b}_t^* (so-called universal portfolio) that perform almost as well as the best portfolio in the sense that for any return sequence $\mathbf{x}_1, \mathbf{x}_2, \dots$, $S_T^* \geq cT^{-\frac{k-1}{2}} S_T(\mathbf{b})$ for all T , where c is a positive constant (depending on k), $S_T^* = \prod_{t=1}^T (\mathbf{b}_t^* \cdot \mathbf{x}_t)$ is the wealth achieved by of the universal portfolio strategy, and $S_T(\mathbf{b}) = \prod_{t=1}^T (\mathbf{b} \cdot \mathbf{x}_t)$ is the wealth achieved by arbitrary constant portfolio \mathbf{b} . The universal portfolio is defined as the mixture $\mathbf{b}_T^* = \int \mathbf{b} P_{T-1}(d\mathbf{b})$, where $P_{T-1}(d\mathbf{b}) = (\prod_{t=1}^{T-1} (\mathbf{b} \cdot \mathbf{x}_t) / Z) D_{1/2}(d\mathbf{b})$, $Z = \int \prod_{t=1}^{T-1} (\mathbf{b} \cdot \mathbf{x}_t) D_{1/2}(d\mathbf{b})$, and $D_{1/2}$ is the $1/2$ -Dirichlet distribution on Γ .

Further development of this approach see in Cover and Ordentlich [9], Vovk and Watkins [20], Blum and Kalai [4], and so on. In this approach the achieved wealth is compared with that of the best in a class of reference strategies. The class of reference strategies considered by Cover [8] is that the class of all constant portfolios defined by all vectors $\mathbf{b} \in \Gamma$. Cover and Ordentlich [9] extended this method for a case where side information in form of states from a finite set can be used by a reference strategy.

The advantage of this worst-case approach is that it avoids imposing statistical models on the stock market and the results hold for all possible sequences $\mathbf{x}_1, \mathbf{x}_2, \dots$. In this sense this approach is extremely robust.

In Section 2, we follow the combined worst-case and stochastic approach. We construct an “artificial probability distribution” of return vectors. This distribution is defined by the well-calibrated forecasts in Dawid [10], [11] sense; these forecasts are constructed using the game-theoretic Blackwell approachability theorem. No stochastic assumptions are made about return vectors for constructing such forecasts. We construct a log-optimal portfolio by scheme (1), where the mathematical expectation E is over the probability distribution defined by the well-calibrated forecasts. Our log-optimal portfolio performs asymptotically at least as well as any stationary portfolio that redistribute the investment at each round using a continuous function of a side information. In Section 3 we discuss a rate of convergence of performance condition.

This approach is not new in the game theory. For example, Foster and Vohra [12], [13] have presented a “calibrated” forecasting scheme which is consistent in hindsight. It allows the agent to choose the best response to the predicted outcome. See also, Mannor and Shimkin [17].

The goal of this paper is to apply these game-theoretic methods to the classical problem of an optimal portfolio construction in order to obtain a further generalization of results presented in the papers [14], [15]. Unlike most previous work we use a more broad reference class of investor strategies – we compare the performance of our portfolio strategy with stationary investor's strategies defined by continuous functions $\mathbf{b}_t = \mathbf{b}(\mathbf{z}_t)$ for all t , where \mathbf{z}_t is a side information.

In Section 4 we show that if $\mathbf{b}(\mathbf{z}_t)$ is allowed to be discontinuous, we cannot prove asymptotic optimality of our portfolio strategy \mathbf{b}_t^* .

2 Main result

Blackwell approachability theorem. We will define our randomized strategy for universal portfolio selection using the Blackwell approachability theorem.

Recall some standard notions of the theory of games. Consider a game between two players with finite sets of their moves (pure strategies) $I = \{s_1, \dots, s_N\}$ and $J = \{a_1, \dots, a_M\}$. A mixed strategy of the first player is a probability distribution on I presented by a vector $\mathbf{P} = (p(1), \dots, p(N))$, where $p(1) + \dots + p(N) = 1$ and $p(i) \geq 0$ for all i . Denote by $P(I)$ a set of all mixed strategies of the first player. Analogously, let $P(J)$ be the set of all mixed strategies of the second player.

Consider an infinitely repeatable game, where at each round t the first player announces a mixed strategy $\mathbf{P}_t \in P(I)$ and the second player announces a pure strategy $j_t \in J$. After that, the first player pick up $i_t \in I$ distributed by \mathbf{P}_t and receives a payoff $f(i_t, j_t)$.

The players can announce their moves independently or, in the adversarial setting, where the second player announces an element j_t after the first player announces \mathbf{P}_t .

By a randomized online strategy of the first player we mean an infinite sequence $\mathbf{P}_1, \mathbf{P}_2, \dots$ of his mixed strategies, where each \mathbf{P}_t is a conditional (with respect to past moves $i_1, j_1, \dots, i_{t-1}, j_{t-1}$) probability distribution on I .

Let a vector-valued payoff function $\mathbf{f}(s_i, a_j) \in \mathcal{R}^d$ be given, where $d \geq 1$, $s_i \in I$, and $a_j \in J$. As usual, $\mathbf{f}(\mathbf{P}, a_j) = \sum_{i=1}^N \mathbf{f}(s_i, a_j)p(i)$ and $\mathbf{f}(\mathbf{P}, \mathbf{Q}) = \sum_{i=1}^N \sum_{j=1}^M \mathbf{f}(s_i, a_j)p(i)q(j)$, where $\mathbf{P} = (p(1), \dots, p(N))$ and $\mathbf{Q} = (q(1), \dots, q(M))$ are elements of $P(I)$ and $P(J)$ respectively.

Note that the Blackwell theorem (see Theorem 1 below) holds for l_2 norm in \mathcal{R}^d . In some special case we also consider l_1 norm $\|\cdot\|_1$. In all other cases we use l_2 norm $\|\cdot\|$. The choice of the norm, l_1 , l_2 or l_∞ , is irrelevant at this stage, since all norms are equivalent on finite-dimensional spaces. The difference takes place in Section 3, where we compute a rate of convergence in Theorem 3.

For any subset $U \subseteq \mathcal{R}^d$ and any vector $\mathbf{x} \in \mathcal{R}^d$, the distance from \mathbf{x} to U is defined $\text{dist}(\mathbf{x}, U) = \inf_{\mathbf{y} \in U} \|\mathbf{x} - \mathbf{y}\|$.

Following Blackwell a set $U \subseteq \mathcal{R}^d$ is called *approachable* if a randomized online strategy $\mathbf{P}_1, \mathbf{P}_2, \dots$ of the first player exists such that

$$\lim_{T \rightarrow \infty} \text{dist} \left(\frac{1}{T} \sum_{t=1}^T \mathbf{f}(i_t, j_t), U \right) = 0$$

holds for P -almost all sequences i_1, i_2, \dots of the first player moves regardless of how the second player chooses j_1, j_2, \dots , where P is an overall probability distribution generated by $\mathbf{P}_1, \mathbf{P}_2, \dots$.

Blackwell [3] proposed a generalization of the minimax theorem for the case of the vector-payoff functions. In particular, he proved the following theorem.

Theorem 1. *A closed convex subset $U \subseteq \mathcal{R}^d$ is approachable by the first player if and only if for every mixed strategy $\mathbf{Q} \in P(J)$ of the second player a mixed strategy $\mathbf{P} \in P(I)$ of the first player exists such that $\mathbf{f}(\mathbf{P}, \mathbf{Q}) \in U$.*

We apply this theorem for the log-optimal portfolio selection.

Optimal portfolio construction. We consider the market process in the game-theoretic framework as a game between two players: *Market* and *Investor*.

In deterministic adversarial setting, the market process is described as follows. At the beginning of each time period t , *Investor*, observing all past moves and a side information \mathbf{z}_t which is an element of some compact metric space C , chooses a portfolio vector \mathbf{b}_t . At the end of this period *Market*, observing all past moves, chooses a return vector \mathbf{x}_t . After that, *Investor* updates his wealth $S_t = S_{t-1} \cdot (\mathbf{b}_t \cdot \mathbf{x}_t)$. For simplicity, we assume that C is some closed interval in \mathcal{R} .

A strategy of *Investor* is called stationary if it is defined by a function from the set of all signals to the simplex of all portfolios: $\mathbf{b}_t = \mathbf{b}(\mathbf{z}_t)$ for all t .

We discretize all basic sets used in the game. For any K , let $\tilde{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_K\}$ be a finite grid in C . For every $\mathbf{z} \in C$ an $\mathbf{c}_j \in \tilde{C}$ exists such that $\|\mathbf{z} - \mathbf{c}_j\| < \nu$, where ν is a level of precision depending on K .

Recall that for any return vector $\mathbf{x} = (x(1), \dots, x(k))$, $x(i) \in [\lambda_1, \lambda_2]$, where λ_1 and λ_2 are real numbers such that $0 < \lambda_1 < \lambda_2$.

For any M , define a finite grid $\tilde{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_M\}$ in the set $[\lambda_1, \lambda_2]^k$ of all return vectors such that for any return vector $\mathbf{a} \in [\lambda_1, \lambda_2]^k$ an element $\mathbf{a}_i \in \tilde{A}$ exists satisfying $\|\mathbf{a} - \mathbf{a}_i\| < \mu$, where $M = (1/\mu)^k$.³

For any vector $\mathbf{a} \in \tilde{A}$, let $\delta[\mathbf{a}] = (0, \dots, 1, \dots, 0)$ be the probability distribution concentrated on element \mathbf{a} of the set \tilde{A} . In this vector of dimension M , the i th coordinate is 1, all other coordinates are 0.

Consider a set $P(\tilde{A}|\tilde{C})$ of all probability distributions on \tilde{A} conditional with respect to elements of the set \tilde{C} . Any such distribution is defined by the KM -dimensional vector $\mathbf{p} = (\mathbf{p}(a_i|c_j) : 1 \leq i \leq M, 1 \leq j \leq K)'$, where $\sum_{i=1}^M \mathbf{p}(a_i|c_j) = 1$ for each $1 \leq j \leq K$, i.e., given $\mathbf{c}_j \in \tilde{C}$, $\mathbf{p}(\cdot|\mathbf{c}_j) = (\mathbf{p}(a_i|c_j) : 1 \leq i \leq M)'$ is an M -dimensional probability vector.⁴

³ In what follows we ignore the problem of rounding.

⁴ \mathbf{x}' is the transposition of a vector \mathbf{x} .

Let $\tilde{P}(\tilde{A}|\tilde{C}) = \{\mathbf{s}_1, \dots, \mathbf{s}_N\}$ be an ϵ -grid in the set $P(\tilde{A}|\tilde{C})$. For any $\mathbf{P} \in P(\tilde{A}|\tilde{C})$ an $\mathbf{s}_i \in \tilde{P}(\tilde{A}|\tilde{C})$ exists such that $\|\mathbf{P} - \mathbf{s}_i\|_1 < \epsilon$.

Now we apply Theorem 1. Consider a two-players infinitely repeated game, where the first player moves are elements of the N -element set $I = \tilde{P}(\tilde{A}|\tilde{C})$ and the second player moves are from the KM -element set $J = \tilde{A} \times \tilde{C}$. At any round t the first player outputs a forecast $\mathbf{p}_t \in \tilde{P}(\tilde{A}|\tilde{C})$ and the second player outputs an outcome $(\mathbf{x}_t, \mathbf{z}_t) \in J$.

Let \mathbf{s}_i be the i th vector of the set $\tilde{P}(\tilde{A}|\tilde{C})$, \mathbf{c}_j be the j th element of the set \tilde{C} , and $\mathbf{a} \in \tilde{A}$, $\mathbf{0}^{KM}$ be the KM -dimensional zero vector. The values of payoff function \mathbf{f} are vectors of dimension KMN :

$$\mathbf{f}(\mathbf{s}_i, (\mathbf{a}, \mathbf{c}_j)) = \begin{pmatrix} \mathbf{0}^{NM} \\ \dots \\ \mathbf{0}^{NM} \\ \mathbf{g}_j(\mathbf{s}_i, (\mathbf{a}, \mathbf{c}_j)) \\ \mathbf{0}^{NM} \\ \dots \\ \mathbf{0}^{NM} \end{pmatrix},$$

where $\mathbf{g}_j(\mathbf{s}_i, \mathbf{a})$ is its j th NM -dimensional column-vector defined as a combination of N column-vectors of dimension M :

$$\mathbf{g}_j(\mathbf{s}_i, (\mathbf{a}, \mathbf{c}_j)) = \begin{pmatrix} \mathbf{0}^M \\ \dots \\ \mathbf{0}^M \\ \delta[\mathbf{a}] - \mathbf{s}_i(\cdot|\mathbf{c}_j) \\ \mathbf{0}^M \\ \dots \\ \mathbf{0}^M \end{pmatrix},$$

where $\delta[\mathbf{a}] - \mathbf{s}_i(\cdot|\mathbf{c}_j)$ is the difference of two M -dimensional column-vectors, which is the i th component of the composite vector $\mathbf{g}_j(\mathbf{s}_i, (\mathbf{a}, \mathbf{c}_j))$.

We now define the convex set $U = \{\mathbf{x} \in \mathcal{R}^{KNM} : \|\mathbf{x}\|_1 \leq \epsilon\}$ in the space \mathcal{R}^{KNM} .

By definition, a randomized strategy of the first player is a sequence $\mathbf{P}_1, \mathbf{P}_2, \dots$, where each \mathbf{P}_t is a conditional (with respect to past moves of both players) probability distribution on $I = \tilde{P}(\tilde{A}|\tilde{C})$.

A set U is approachable if a randomized strategy $\mathbf{P}_1, \mathbf{P}_2, \dots$ of the first player exists such that $\lim_{t \rightarrow \infty} \text{dist}(\mathbf{f}(\mathbf{p}_t, (\mathbf{x}_t, \mathbf{z}_t)), U) = 0$ almost surely regardless of the second player moves $(\mathbf{x}_1, \mathbf{z}_1), (\mathbf{x}_2, \mathbf{z}_2), \dots$, where the trajectory $\mathbf{p}_1, \mathbf{p}_2, \dots$ is distributed according to the overall probability distribution P and $\mathbf{p}_t \in \tilde{P}(\tilde{A}|\tilde{C})$ for all t .

By Theorem 1 the closed convex set U is approachable if and only if for each $\mathbf{Q} \in \mathcal{P}(\tilde{A} \times \tilde{C})$ an $\mathbf{P} \in \mathcal{P}(\tilde{P}(\tilde{A}|\tilde{C}))$ exists such that $\mathbf{f}(\mathbf{P}, \mathbf{Q}) \in U$.

Let $\mathbf{Q}'(\mathbf{c}_j) = \sum_{i=1}^M \mathbf{Q}(\mathbf{a}_i, \mathbf{c}_j)$ be the marginal probability distribution on \tilde{C} and $\mathbf{Q}''(\mathbf{a}_i|\mathbf{c}_j) = \mathbf{Q}(\mathbf{a}_i, \mathbf{c}_j)/\mathbf{Q}'(\mathbf{c}_j)$ be the conditional probability, where $1 \leq j \leq K$.

Let $\mathbf{s}_k \in \tilde{P}(\tilde{A}|\tilde{C})$ be such that $\|\mathbf{s}_k - \mathbf{Q}''\|_1 < \epsilon$ and $\mathbf{P} = \delta[\mathbf{s}_k]$ be the probability distribution on $\tilde{P}(\tilde{A}|\tilde{C})$ concentrated on \mathbf{s}_k . Then by definition

$$\mathbf{f}(\mathbf{P}, \mathbf{Q}) = (\mathbf{g}_1(\mathbf{P}, \mathbf{Q}), \dots, \mathbf{g}_j(\mathbf{P}, \mathbf{Q}), \dots, \mathbf{g}_K(\mathbf{P}, \mathbf{Q}))',$$

where

$$\begin{aligned} \mathbf{g}_j(\mathbf{P}, \mathbf{Q}) &= \begin{pmatrix} \mathbf{0}^M \\ \dots \\ \mathbf{0}^M \\ \mathbf{Q}(\mathbf{a}_1, \mathbf{c}_j) - \mathbf{s}_k(\mathbf{a}_1|\mathbf{c}_j) \sum_{i=1}^M Q(\mathbf{a}_i, \mathbf{c}_j) \\ \dots \\ \mathbf{Q}(\mathbf{a}_i, \mathbf{c}_j) - \mathbf{s}_k(\mathbf{a}_i|\mathbf{c}_j) \sum_{i=1}^M Q(\mathbf{a}_i, \mathbf{c}_j) \\ \dots \\ \mathbf{Q}(\mathbf{a}_M, \mathbf{c}_j) - \mathbf{s}_k(\mathbf{a}_M|\mathbf{c}_j) \sum_{i=1}^M Q(\mathbf{a}_i, \mathbf{c}_j) \\ \mathbf{0}^M \\ \dots \\ \mathbf{0}^M \end{pmatrix} = \\ &= \\ &= Q'(\mathbf{c}_j) \begin{pmatrix} \mathbf{0}^M \\ \dots \\ \mathbf{0}^M \\ \mathbf{Q}''(\mathbf{a}_1|\mathbf{c}_j) - \mathbf{s}_k(\mathbf{a}_1|\mathbf{c}_j) \\ \dots \\ \mathbf{Q}''(\mathbf{a}_i|\mathbf{c}_j) - \mathbf{s}_k(\mathbf{a}_i|\mathbf{c}_j) \\ \dots \\ \mathbf{Q}''(\mathbf{a}_M|\mathbf{c}_j) - \mathbf{s}_k(\mathbf{a}_M|\mathbf{c}_j) \\ \mathbf{0}^M \\ \dots \\ \mathbf{0}^M \end{pmatrix} \end{aligned}$$

for $1 \leq j \leq K$. From this $\|\mathbf{f}(\mathbf{P}, \mathbf{Q})\|_1 = \sum_{j=1}^K \mathbf{Q}'(\mathbf{c}_j) \|\mathbf{s}_k(\cdot|\mathbf{c}_j) - Q''(\cdot|\mathbf{c}_j)\|_1 < \epsilon$ follows and then $\mathbf{f}(\mathbf{P}, \mathbf{Q}) \in U$.

By a randomized online strategy of *Investor* (first player) we mean a sequence of conditional probability distributions $\mathbf{P}_t = \mathbf{P}_t(\mathbf{s}|\sigma_t, \mathbf{z}_t)$, $t = 1, 2, \dots$, on $\tilde{P}(\tilde{A}|\tilde{C})$, where $\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})$, $\sigma_t = (\mathbf{z}_1, \mathbf{p}_1, \mathbf{x}_1, \dots, \mathbf{z}_{t-1}, \mathbf{p}_{t-1}, \mathbf{x}_{t-1})$ is a history, and $\mathbf{z}_t \in \tilde{C}$.

By Theorem 1 a randomized strategy $\mathbf{P}_1, \mathbf{P}_2, \dots$ of the first player exists, such that regardless of that sequence $(\mathbf{x}_1, \mathbf{z}_1), (\mathbf{x}_2, \mathbf{z}_2), \dots$ was announced by the

second player the sequence of the vector-valued payoffs

$$\bar{\mathbf{n}}_t = \frac{1}{T} \sum_{t=1}^T \mathbf{f}(\mathbf{p}_t, (\mathbf{x}_t, \mathbf{z}_t)) = \begin{pmatrix} \frac{1}{T} \sum_{t=1}^T I_{\{\mathbf{p}_t=\mathbf{s}_1, \mathbf{z}_t=\mathbf{c}_1\}} (\delta[\mathbf{x}_t] - \mathbf{s}_1(\cdot|\mathbf{c}_1)) & \dots & \frac{1}{T} \sum_{t=1}^T I_{\{\mathbf{p}_t=\mathbf{s}_N, \mathbf{z}_t=\mathbf{c}_1\}} (\delta[\mathbf{x}_t] - \mathbf{s}_N(\cdot|\mathbf{c}_1)) & \dots & \frac{1}{T} \sum_{t=1}^T I_{\{\mathbf{p}_t=\mathbf{s}_1, \mathbf{z}_t=\mathbf{c}_K\}} (\delta[\mathbf{x}_t] - \mathbf{s}_1(\cdot|\mathbf{c}_K)) & \dots & \frac{1}{T} \sum_{t=1}^T I_{\{\mathbf{p}_t=\mathbf{s}_N, \mathbf{z}_t=\mathbf{c}_K\}} (\delta[\mathbf{x}_t] - \mathbf{s}_N(\cdot|\mathbf{c}_K)) \end{pmatrix}.$$

P -almost surely approaches the set U , where P is an overall probability distribution generated by a the sequence $\mathbf{P}_1, \mathbf{P}_2, \dots$ of conditional distributions and the trajectory $\mathbf{p}_1, \mathbf{p}_2, \dots$ is distributed by the measure P ; by $I_{\{\mathbf{p}_t=\mathbf{s}_1, \mathbf{z}_t=\mathbf{c}_j\}}$ we denote the indicator function.

Let $\mathbf{p}_1, \mathbf{p}_2, \dots$ be a sequence of the well-calibrated forecasts distributed according to $\mathbf{P}_1, \mathbf{P}_2, \dots$. Denote

$$N_T(\mathbf{s}, i, j) = |\{t : \mathbf{p}_t = \mathbf{s}, 1 \leq t \leq T, \mathbf{x}_t = \mathbf{a}_i, \mathbf{z}_t = \mathbf{c}_j\}|$$

and

$$M_T(\mathbf{s}, j) = |\{t : \mathbf{p}_t = \mathbf{s}, 1 \leq t \leq T, \mathbf{z}_t = \mathbf{c}_j\}|,$$

where $1 \leq i \leq M$, $1 \leq j \leq K$, and \mathbf{s} be an arbitrary element of the grid $\tilde{P}(\tilde{A}|\tilde{C})$.

Approachability of the set U implies the following theorem which is an immediate corollary of Theorem 1.

Theorem 2. *There exists a randomized online strategy $\mathbf{P}_1, \mathbf{P}_2, \dots$ such that for any sequence $\mathbf{z}_1, \mathbf{x}_1, \dots, \mathbf{z}_t, \mathbf{x}_t, \dots$*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} \sum_{1 \leq i \leq M} |N_T(\mathbf{s}, i, j) - M_T(\mathbf{s}, j)s(i|\mathbf{c}_j)| \leq \epsilon \quad (3)$$

for almost all sequences $\mathbf{p}_1, \mathbf{p}_2, \dots$ distributed according to the overall probability distribution generated by $\mathbf{P}_1, \mathbf{P}_2, \dots$, where \mathbf{s} is an arbitrary element of $\tilde{P}(\tilde{A}|\tilde{C}) = \{\mathbf{s}_1, \dots, \mathbf{s}_M\}$.

We call forecasts $\mathbf{p}_1, \mathbf{p}_2, \dots$ satisfying (3) ϵ -calibrated. If (3) holds for each $\epsilon > 0$ then these forecasts are called well-calibrated.

A difference with Foster–Vohra [13] calibration is that each \mathbf{p}_t is a conditional probability distribution and that this is calibration with respect to a side information.

Let $\mathbf{c}_j \in \tilde{C}$ and \mathbf{X} be a random variable distributed according to the probability distribution $\mathbf{s}(\cdot|\mathbf{c}_j) = (s(1|\mathbf{c}_j), \dots, s(M|\mathbf{c}_j))$, where $P\{\mathbf{X} = \mathbf{a}_i\} = s(i|\mathbf{c}_j)$

for every $1 \leq i \leq M$. For any probability distribution $\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C}) = \{\mathbf{s}_1, \dots, \mathbf{s}_N\}$ and any $\mathbf{c}_j \in \tilde{C}$ define the optimal portfolio

$$\mathbf{b}^*(\mathbf{s}|\mathbf{c}_j) = \operatorname{argmax}_{\mathbf{b}} E_{\mathbf{X} \sim \mathbf{s}(\cdot|\mathbf{c}_j)}(\log(\mathbf{b} \cdot \mathbf{X})). \quad (4)$$

We can rewrite (4) also as $\mathbf{b}^*(\mathbf{s}|\mathbf{c}_j) = \operatorname{argmax}_{\mathbf{b}} \sum_{i=1}^M (\log(\mathbf{b} \cdot \mathbf{a}_i)) s(i|\mathbf{c}_j)$.

Let $\tilde{B}_t = \{\mathbf{b}^*(\mathbf{s}_1|\mathbf{z}_t), \dots, \mathbf{b}^*(\mathbf{s}_N|\mathbf{z}_t)\}$ be the set of all optimal portfolios, where $\mathbf{z}_t \in \tilde{C}$ be a side information at round t . Using a randomized forecast $\mathbf{p}_t \in \tilde{P}(\tilde{A}|\tilde{C})$ distributed according to \mathbf{P}_t existing by the Blackwell approachability theorem, we can define at any round t of the game presented on Fig 1 the random portfolio

$$\mathbf{b}_t^* = \mathbf{b}^*(\mathbf{p}_t|\mathbf{z}_t) = \operatorname{argmax}_{\mathbf{b}} E_{\mathbf{X} \sim \mathbf{p}_t(\cdot|\mathbf{z}_t)}(\log(\mathbf{b} \cdot \mathbf{X})), \quad (5)$$

where $\mathbf{p}_t \in \tilde{P}(\tilde{A}|\tilde{C})$ is the random forecast announced at round t and \mathbf{z}_t is the signal at step t .

The corresponding randomized algorithm is presented on Fig. 1. We consider two types of investors: *Investor* uses the randomized algorithm for computing an optimal portfolio, *Stationary Investor* uses a continuous function of the side information.

```

FOR  $t = 1, 2, \dots$ 
  Market announces a signal  $\mathbf{z}_t$ .
  Investor announces a probability distribution  $\mathbf{P}_t = \mathbf{P}_t(\cdot|\sigma_t, \mathbf{z}_t)$  on the set  $\tilde{P}(\tilde{A}|\tilde{C})$ .
  Market announces a return vector  $\mathbf{x}_t$ .
  Investor pick up a portfolio  $\mathbf{b}_t^* \in \tilde{B}_t$  distributed by  $\mathbf{P}_t$  considered on  $\tilde{B}_t$  and updates
  his wealth  $S_t^* = S_{t-1}^* \cdot (\mathbf{b}_t^* \cdot \mathbf{x}_t)$ .
  Stationary Investor updates his wealth  $S_t = S_{t-1} \cdot (\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)$ .
ENDFOR

```

Fig. 1. Protocol of portfolio game

We have presented the construction using the approximation grids of a fixed accuracy. The complete construction is divided on time intervals $1 \leq t_1 \leq \dots < t_n \leq \dots$. At time steps $t \in [t_n, t_{n+1})$ the grids \tilde{C}_n , \tilde{A}_n of cardinality K_n , M_n and $\tilde{P}_n(\tilde{A}_n|\tilde{C}_n)$ of cardinality N_n are used, where $n = 1, 2, \dots$. These grids approximate the sets C , A , and $P(A)$ with an increasing degree of accuracy: $M_n \rightarrow \infty$, $K_n \rightarrow \infty$ and $\epsilon_n \rightarrow 0$, $\mu_n \rightarrow 0$ as $n \rightarrow \infty$.

The following theorem asserts that in this case portfolio (5) is almost surely log-optimal with respect to the class of all portfolios presented by continuous functions $\mathbf{b}(\cdot) : C \rightarrow \mathcal{R}$.

Theorem 3. *The randomized portfolio strategy \mathbf{b}_t^* defined by (5) and by refining the discretization incrementally is almost surely log-optimal for the class of all*

continuous portfolio strategies $\mathbf{b}(\cdot)$:

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq 0 \quad (6)$$

for almost all trajectories $\mathbf{p}_1, \mathbf{p}_2, \dots$, where $S_T^* = \prod_{t=1}^T (\mathbf{b}_t^* \cdot \mathbf{x}_t)$ is the wealth achieved by of the universal portfolio strategy, $S_T = \prod_{t=1}^T (\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)$ is the wealth achieved by an arbitrary portfolio $\mathbf{b}(\mathbf{z}_t)$, and \mathbf{z}_t is a signal at any round t .

Proof. The complete proof is based on the construction which is divided on time intervals $1 \leq t_1 \leq \dots < t_n \leq \dots$ and the grids with increasing degree of accuracy as indicated above. For simplicity of presentation, we give the proof only for one of such grids. Given $\epsilon > 0$ and $\mu > 0$, we consider the corresponding grids \tilde{C} , \tilde{A} , and $\tilde{P}(\tilde{A}|\tilde{C})$ and prove optimality of the universal portfolio up to $O(\mu + \epsilon)$. Notice that $M = (1/\mu)^k$ and $N = (1/\epsilon)^{KM-1}$.

Replace the sums with return vectors and signals on their approximations from the corresponding grids. Let us estimate the loss of accuracy as a result of such replacements.

Notice that for any $\mathbf{b} \in \Gamma$, $\mathbf{x} \in A$ and $\mathbf{a} \in \tilde{A}$ such that $\|\mathbf{x} - \mathbf{a}\| < \mu$ we have $|(\mathbf{b} \cdot \mathbf{x}) - (\mathbf{b} \cdot \mathbf{a})| = |(\mathbf{b} \cdot (\mathbf{x} - \mathbf{a}))| \leq \|\mathbf{b}\| \|\mathbf{x} - \mathbf{a}\| \leq \|\mathbf{x} - \mathbf{a}\| < \mu$. Then for $\mathbf{b} \cdot \mathbf{x} \geq (\mathbf{b} \cdot \mathbf{a})$, $|\ln(\mathbf{b} \cdot \mathbf{x}) - \ln(\mathbf{b} \cdot \mathbf{a})| = \left| \ln \frac{(\mathbf{b} \cdot \mathbf{x})}{(\mathbf{b} \cdot \mathbf{a})} \right| = \left| \ln \left(1 + \frac{(\mathbf{b} \cdot \mathbf{x}) - (\mathbf{b} \cdot \mathbf{a})}{(\mathbf{b} \cdot \mathbf{a})} \right) \right| \leq \left| \frac{(\mathbf{b} \cdot \mathbf{x}) - (\mathbf{b} \cdot \mathbf{a})}{(\mathbf{b} \cdot \mathbf{a})} \right| \leq \mu / \lambda_1$, where \ln is the natural logarithm; if $\mathbf{b} \cdot \mathbf{x} < (\mathbf{b} \cdot \mathbf{a})$ we can exchange role of \mathbf{a} and \mathbf{x} .

Let $\mathbf{b}(\cdot)$ be an arbitrary continuous stationary portfolio strategy. Given $\epsilon > 0$ consider a sufficiently accurate approximating grid $\tilde{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_K\}$ in the set C of all signals satisfying the following property: for each $\mathbf{z} \in C$ an $\mathbf{c}_i \in \tilde{C}$ exists such that⁵ $\|\mathbf{b}(\mathbf{z}) - \mathbf{b}(\mathbf{c}_i)\| < \epsilon$.

Therefore, all sums are changed on $O(\mu + \epsilon)$ as a result of these replacements of return vectors and signals. Assuming that return vectors \mathbf{x}_t and signals \mathbf{z}_t are now elements of the corresponding finite grids and using continuity of the function $\mathbf{b}(\cdot)$, we obtain the estimate for the mean wealth of an arbitrary portfolio $\mathbf{b}(\cdot)$:

$$\begin{aligned} \frac{1}{T} \log S_T &= \frac{1}{T} \sum_{t=1}^T \log(\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t) = \\ &= \sum_{j=1}^K \frac{1}{T} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} \sum_{i=1}^M N_T(\mathbf{s}, i, j) \log(\mathbf{b}(\mathbf{c}_j) \cdot \mathbf{a}_i) + O(\epsilon + \mu), \end{aligned} \quad (7)$$

⁵ Since the complete construction is based on the sequence of ϵ_k -grids, where $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, for each continuous function $\mathbf{b}(\cdot)$ an ϵ_k -grid exists such that this property holds.

where $N_T(\mathbf{s}, i, j) = |\{t : \mathbf{p}_t = \mathbf{s}, 0 \leq t \leq T, \mathbf{x}_t = \mathbf{a}_i, \mathbf{z}_t = \mathbf{c}_j\}|$. By (3) of Theorem 2

$$\frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} \sum_{1 \leq i \leq M} |N_T(\mathbf{s}, i, j) - M_T(\mathbf{s}, j)s(i|\mathbf{c}_j)| \leq \epsilon + o(1)$$

as $T \rightarrow \infty$ almost surely.

Starting from (7) we obtain the following chain of equalities and inequalities which are valid almost surely

$$\begin{aligned} & \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} \sum_{i=1}^M N_T(\mathbf{s}, i, j) \log(\mathbf{b}(\mathbf{c}_j) \cdot \mathbf{a}_i) = \\ & \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} M_T(\mathbf{s}, j) \sum_{i=1}^M \log(\mathbf{b}(\mathbf{c}_j) \cdot \mathbf{a}_i) s(i|\mathbf{c}_j) + O(\epsilon) + o(1) = \\ & \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} M_T(\mathbf{s}, j) E_{\mathbf{X} \sim \mathbf{s}(\cdot|\mathbf{c}_j)} (\log(\mathbf{b}(\mathbf{c}_j) \cdot \mathbf{X})) + O(\epsilon) + o(1) \leq \\ & \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} M_T(\mathbf{s}, j) E_{\mathbf{X} \sim (\mathbf{s}(\cdot|\mathbf{c}_j))} (\log(\mathbf{b}^*(\mathbf{s}|\mathbf{c}_j) \cdot \mathbf{X})) + O(\epsilon) + o(1) = \\ & \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} M_T(\mathbf{s}, j) \sum_{i=1}^M \log(\mathbf{b}^*(\mathbf{s}|\mathbf{c}_j) \cdot \mathbf{a}_i) s(i|\mathbf{c}_j) + O(\epsilon) + o(1) = \\ & \frac{1}{T} \sum_{1 \leq j \leq K} \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} \frac{1}{T} \sum_{i=1}^M N_T(\mathbf{s}, i, j) \log(\mathbf{b}^*(\mathbf{s}|\mathbf{c}_j) \cdot \mathbf{a}_i) + O(\epsilon) + o(1). \quad (8) \end{aligned}$$

as $T \rightarrow \infty$.

Now, we change from (8) to general setting

$$\begin{aligned} & \sum_{j=1}^K \sum_{\mathbf{s} \in \tilde{P}(\tilde{A}|\tilde{C})} \frac{1}{T} \sum_{i=1}^M N_T(\mathbf{s}, i, j) \log(\mathbf{b}^*(\mathbf{s}|\mathbf{c}_j) \cdot \mathbf{a}_i) + O(\mu + \epsilon) + o(1) = \\ & \frac{1}{T} \sum_{t=1}^T \log(\mathbf{b}_t^* \cdot \mathbf{x}_t) + O(\mu + \epsilon) + o(1) = \frac{1}{T} \log S_T^* + O(\mu + \epsilon) + o(1) \quad (9) \end{aligned}$$

almost surely, where S_T^* is the wealth achieved by the optimal portfolio.

Relations (7) and (9) imply that almost surely $\liminf_{T \rightarrow \infty} \frac{1}{T} \log \frac{S_T^*}{S_T} \geq -c(\epsilon + \mu)$, where c is a positive constant. The proof for the case of a fixed approximating grid is complete.

3 Some remarks on rate of convergence

In this section following Mannor and Stoltz [18] we discuss rate of convergence in the optimality condition (6). This rate is defined by the rate of convergence of

a calibrated forecaster in the Blackwell approachability theorem (3) and on the infinite series of grids \tilde{C} , \tilde{A} , and $\tilde{P}(\tilde{A}|\tilde{C})$. We assume that all portfolios functions $\mathbf{b}(\cdot)$ are Lipschitz continuous.

The proof of the approachability theorem gives rise to an implicit strategy, as indicated in Blackwell [3]. We start from a variant the Blackwell theorem for l_2 norm $\|\cdot\|_2$. Denote $d_U(\mathbf{x})$ the projection of \mathbf{x} in l_2 -norm onto U . According to the proof of this theorem (see Blackwell [3] or Cesa-Bianchi and Lugosi [7], Section 7), at each round $t > 2$ and with the notations above, the forecaster should pick his action \mathbf{p}_t at random according to a distribution \mathbf{P}_t on the set $\tilde{P}(\tilde{A}|\tilde{C})$ such that $(\bar{m}_{t-1} - d_U(\bar{m}_{t-1})) \cdot (\mathbf{f}(\mathbf{P}_t, (\mathbf{a}_j, \mathbf{c}_k)) - \bar{m}_{t-1}) \leq 0$ for all $1 \leq j \leq M$ and $1 \leq k \leq K$.

Proof of the Blackwell theorem from Blackwell [3] and convergence theorem for Hilbert space-valued martingales of Kallenberg and Sztencel [16] (see also Chen and White [6]) provide uniform convergence rates of sequence of empirical payoff vectors \bar{m}_t to the target set U : there exists an absolute constant c such that for any $\delta > 0$ for all strategies of *Market* and for all T , with probability $1 - \delta$,

$$\|\bar{m}_t - d_U(\bar{m}_t)\|_2 \leq c \sqrt{\frac{\log \frac{1}{\delta}}{T}}, \quad (10)$$

where $\|\cdot\|_2$ is the Euclidian norm in R^{KNM} .⁶

The set U used in the proof of Theorem 3 is defined using l_1 -norm. Then, using triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\|\bar{m}_T\|_1 \leq \|d_U(\bar{m}_T)\|_1 + \|\bar{m}_T - d_U(\bar{m}_T)\|_1 \leq \epsilon + \sqrt{KMN} \|\bar{m}_T - d_U(\bar{m}_T)\|_2,$$

where $N = O((1/\epsilon)^{KM-1})$. Hence, by (10) we have

$$\|\bar{m}_T\|_1 \leq c\epsilon + c \sqrt{\frac{\log \frac{1}{\delta}}{\epsilon^{KM-1}T}}. \quad (11)$$

The suitable choice of ϵ is when both terms of the sum (11) are of the same order of magnitude, i.e., $\epsilon \sim T^{-\frac{1}{KM+1}}$. This is the optimal level of precision in (8).

Further, we have to find the optimal level of precision in (7). We should optimize the choice of $M = (1/\mu)^k$, where μ is the level of precision of approximating the values \mathbf{x}_t .

Assuming that the signal space C is a closed interval of real numbers, $\mathbf{b}(\cdot)$ is Lipschitz continuous, and getting $K = 1/\mu$, we should choose the suitable μ to minimize the sum $\mu + T^{-\frac{1}{KM+1}}$, where $KM = (1/\mu)^{k+1}$. The optimal value is $\mu \sim (\log T)^{-\frac{1}{k+2}}$.

Combining series of grids, like it was done in in Vyugin [21], we can obtain a rough bound $O(\log T)^{-\frac{1}{k+2}+\nu}$ of the rate of convergence in (6), where ν is an

⁶ See Cesa-Bianchi and Lugosi [7], Exersice 7.23. Recall that $N = |\tilde{P}(\tilde{A}|\tilde{C})|$, $M = |\tilde{A}|$, and $K = |\tilde{C}|$.

arbitrary small positive real number, and k is the number of assets. More precise, for any $\delta > 0$, with probability $1 - \delta$,

$$\frac{1}{T} \log S_T^* \geq \frac{1}{T} \log S_T - c(\log(T/\log \frac{1}{\delta}))^{-\frac{1}{k+2}+\nu}$$

for all T , where c is a constant and $S_T = \prod_{t=1}^T (\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)$ is the wealth achieved by an arbitrary Lipschitz continuous portfolio $\mathbf{b}(\cdot)$.

4 Competing with discontinuous stationary strategies

The portfolio strategy \mathbf{b}_t^* defined in Theorem 3 performs asymptotically at least as well as any continuous strategy $\mathbf{b}(\mathbf{z}_t)$. A weak point the strategy $\mathbf{b}(\mathbf{z}_t)$ is that a continuous function cannot respond sufficiently quickly to information about changes of the return vectors.

A positive argument in favor of the requirement of continuity of $\mathbf{b}(\cdot)$ is that it is natural to compete only with computable trading strategies, and continuity is often regarded as a necessary condition for computability (Brouwer's "continuity principle").

If $\mathbf{b}(\cdot)$ is allowed to be discontinuous, we cannot prove asymptotic optimality of our portfolio strategy \mathbf{b}_t^* . We present below the corresponding construction.

Consider a portfolio game with two assets and suppose that an algorithm exists which when fed with t and a history $\sigma_t = \mathbf{b}_1^*, \mathbf{x}_1, \dots, \mathbf{b}_{t-1}^*, \mathbf{x}_{t-1}$ outputs a probability distribution $P_t = P_t(\cdot | \sigma_t)$ on the simplex of all portfolios and let a portfolio \mathbf{b}_t^* be distributed by P_t . Denote by $\mathbf{e}_t = E_{P_t}(\mathbf{b}_t^*) = (e_{1,t}, e_{2,t})'$ the conditional mathematical expectation of \mathbf{b}_t^* with respect to σ_t .

Suppose that $\mathbf{z}_t = \sigma_t$ be a signal at round t . Define $\mathbf{b}(\mathbf{z}_t) = (b_1(\mathbf{z}_t), b_2(\mathbf{z}_t))'$, where

$$b_1(\mathbf{z}_t) = \begin{cases} 1 & \text{if } e_{1,t} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and $b_2(\mathbf{z}_t) = 1 - b_1(\mathbf{z}_t)$.

For any t , define a return vector $\mathbf{x}_t = (x_{1,t}, x_{2,t})'$, where $x_{1,t} = 2$, $x_{2,t} = 1$ if $e_{1,t} \leq \frac{1}{2}$ and $x_{1,t} = 1$, $x_{2,t} = 2$ otherwise.

Let $S_T = \prod_{t=1}^T (\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)$ be a wealth achieved by the stationary portfolio $\mathbf{b}(\cdot)$ and $S_T^* = \prod_{t=1}^T (\mathbf{b}_t^* \cdot \mathbf{x}_t)$ be a wealth achieved by the randomized portfolio strategy \mathbf{b}_t^* in the first T rounds.

By definition $\log \frac{S_T}{S_T^*} = \sum_{t=1}^T \log \frac{(\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)}{(\mathbf{b}_t^* \cdot \mathbf{x}_t)}$. It is easy to verify that for any t ,

$$\log \frac{(\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)}{(\mathbf{b}_t^* \cdot \mathbf{x}_t)} = \begin{cases} \frac{2}{1+b_{1,t}^*} & \text{if } e_{1,t} \leq \frac{1}{2} \\ \frac{2}{1+b_{2,t}^*} & \text{otherwise} \end{cases}$$

Then for any t ,

$$E_{P_t} \left(\log \frac{(\mathbf{b}(\mathbf{z}_t) \cdot \mathbf{x}_t)}{(\mathbf{b}_t^* \cdot \mathbf{x}_t)} \right) = \begin{cases} E_{P_t} \frac{2}{1+b_{1,t}^*} \geq \frac{1}{2} E_{P_t} \frac{1-b_{1,t}^*}{1+b_{1,t}^*} \geq \frac{1}{8} & \text{if } e_{1,t} \leq \frac{1}{2} \\ E_{P_t} \frac{2}{1+b_{2,t}^*} \geq \frac{1}{2} E_{P_t} \frac{1-b_{2,t}^*}{1+b_{2,t}^*} \geq \frac{1}{8} & \text{otherwise} \end{cases}$$

Then $E_P \left(\log \frac{S_T}{S_T^*} \right) \geq \frac{1}{8}T$ for all T , where P is an overall probability distribution defined by all P_t , $t = 1, 2, \dots$. By the martingale law of large numbers

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \frac{S_T}{S_T^*} \geq \frac{1}{8}$$

almost surely. Therefore, the stationary portfolio strategy $\mathbf{b}(\cdot)$ outperforms the portfolio strategy \mathbf{b}_t^* almost surely.

5 Conclusion

In this paper we show how the game-theoretic methods can be applied to the classical problems of a universal portfolio construction. We present the method for constructing a log-optimal portfolio in a game-theoretic framework and in adversarial setting. No stochastic assumptions are made about return vectors. Instead, we define “an artificial probability distribution” for return vectors using the method of calibration. Using this distribution, we construct the log-optimal portfolio by the standard scheme (1), where the mathematical expectation E is over probability distribution defined by well-calibrated forecasts. Our log-optimal portfolio performs asymptotically at least as well as any stationary portfolio that redistribute the investment at each round using a continuous function of the side information. This performance is almost surely, where the corresponding probability distribution is an internal distribution of the probabilistic algorithm computing well-calibrated forecasts on the base of the Blackwell approachability theorem. Theorem 3 is valid not only for log-loss function but also for any Lipschitz continuous loss-function.

The drawback of this approach is the very poor rate of convergence in (6), but it is true for basically all nontrivial applications of approachability (probably except of V’yugin [21]) because the constructions used to employ it results in many extra dimensions which do not suit l_2 geometry for the problem. Note that no rate of convergence exists for portfolio strategies universal with respect to the class of all stationary and ergodic processes.

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